Worcester County Mathematics League

Varsity Meet 1
October 3, 2012

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Varsity Meet 1 – October 3, 2012
Round 1: Arithmetic

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Evaluate: $2013^2 + 2012^2 - 2(2012)(2013)$.

2. Let the operation $x♠y$ be defined as $\dfrac{x+y}{xy}$. Given that $a♠(2♠3) = 4♠1$, find the value of a.

3. Evaluate the following:

$$-|-3|^2 + (-1)^{-12} \div \left(\sqrt{1}+1\right) \cdot 2^2 + \frac{6(2+1)}{\sqrt[3]{27}}$$

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 1 – October 3, 2012
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Three measurements are recorded in an experiment. Two are measured as 64 and 78. The average is 80. What is the third measurement?

2. Simplify:

$$\frac{\frac{p}{q} - \frac{p-q}{p+q}}{\frac{q}{p} + \frac{p-q}{p+q}}$$

3. Given that $x + \frac{1}{x} = 5$, find $x^3 + \frac{1}{x^3}$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 1 – October 3, 2012
Round 3: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Let $A = \{\text{multiples of 3 between 1 and 100}\}$ and let $B = \{\text{multiples of 7 between 1 and 100}\}$. How many elements are in the intersection $A \cap B$?

2. Of the 400 students at Hogwarts, 150 report that they are comfortable with Transfiguration, 200 say they are comfortable with Potions, and 80 say they are not comfortable with either subject. Based on this information, how many students feel comfortable with both subjects?

3. How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 – October 3, 2012
Round 4: Measurement

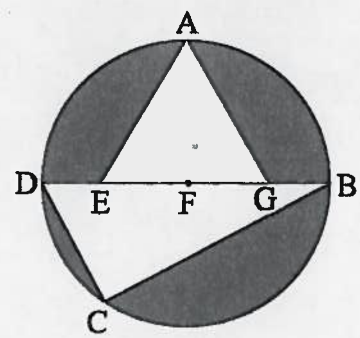
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Find the area of the largest triangle that can be inscribed in a semicircle with radius 2.

2. The radius of a cylinder is 14 units and the height is 5 units. A positive number can be added to either the radius or the height to give the same increase in volume. What is this number?

3. In circle F with radius r, $DC = r$ and $\triangle AEG$ is equilateral with $\overline{AF}$ as an altitude. If the area of the shaded region can be written as $\dfrac{P\pi r^2 + Qr^2\sqrt{3}}{6}$, find the value of $P + Q$.



ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. ________________

(3 pts.) 3. ________________

Varsity Meet 1 – October 3, 2012
Round 5: Polynomial Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. When written in standard form, what is the sum of the coefficients of the quintic polynomial $(13x - 15)^5$?

2. One root of $5x^2 + 34x = k$ is -8. Find the other root.

3. If the roots of $5x^2 - 13x + 2012 = 0$ are m and n, find the quadratic polynomial with roots $m-3$ and $n-3$ and write it in the form $ax^2+bx+c = 0$ with a, b, c relatively prime integers and $a > 0$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 1 – October 3, 2012
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. Let $x \flat y = x^2 - 2xy + y^2$ and let $x \sharp y = x^2 + 2xy + y^2$. Find the value of
$$\frac{(a \sharp b) \flat (d \sharp c)}{b \flat d}$$
if $a = 3$, $b = -2$, $c = -1$, and $d = 2$.

2. A driver averaged 42 mph traveling to work. At what average speed must she return the same distance in order to average 48 mph for the round trip?

3. The universal set for this problem is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Given that $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, and S' denotes the complement of set S, find
$$[[(A' \cup B') \cap (A \cup B)] \cup A]'.$$

4. A right circular cone with a base of radius 1 is inscribed in a sphere of radius 2 such that its volume is maximal. What is the ratio of the volume of the cone to the volume of the sphere? Express your answer as a fraction.

5. Find all possible values of x that satisfy the equation
$$-11x + 6x\sqrt{x+1} + 9\sqrt{x+1} = 11.$$

6. Express the product $(0.3\overline{6}) \times (0.\overline{15})$ as a fraction in lowest terms.

7. Quadrilateral $ABCD$ is inscribed in a circle. If $m\angle A = (6x + 32)^\circ$, $m\angle B = (10x + 5)^\circ$, and $m\angle C = (3x + 40)^\circ$, find the degree-measure of angle D.

8. In a race, if Christina's running speed was 3/4 of Angela's and Moira's speed was 2/3 of Christina's, then Angela's speed was how many times the average of the other two runners' speeds?

9. Let S be the set of all positive integers that leave a remainder of 4 when divided by 7 and a remainder of 2 when divided by 11. What is the smallest element of S?

Varsity Meet 1 – October 3, 2012
TEAM ROUND ANSWER SHEET

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

1. ______________________

2. ______________________ mph

3. { ______________________ }

4. ______________________

5. ______________________

6. ______________________

7. ______________________ °

8. ______________________

9. ______________________

Varsity Meet 1 – October 3, 2012
ANSWERS

ROUND 1
(Leicester, Doherty, Tahanto)

1. 1
2. 20
3. -1

ROUND 2
(Bromfield, Notre Dame, QSC)

1. 98
2. p/q
3. 110

ROUND 3
(Shepherd Hill, Hudson, QSC)

1. 4
2. 30
3. 480

ROUND 4
(Notre Dame, Hudson, Worc Academy)

1. 4
2. $56/5 = 11\frac{1}{5} = 11.2$
3. 1

ROUND 5
(QSC, Auburn, QSC)

1. -32
2. $6/5 = 1\frac{1}{5} = 1.2$
3. $5x^2 + 17x + 2018 = 0$ (need the $= 0$)

TEAM ROUND
(Shepherd Hill, Worcester Academy, Quaboag, Algonquin, QSC, Quaboag, Hudson, Hudson, QSC)

1. 0
2. 56
3. $\{5, 7, 9\}$ (in any order)
4. $\dfrac{2+\sqrt{3}}{32} \approx 0.117$
5. $-1, -8/9, 5/4$ (need all 3, in any order)
6. $1/18$ (must be a fraction)
7. $55°$
8. $8/5 = 1\frac{3}{5} = 1.6$
9. 46

Varsity Meet 1 – October 3, 2012
FULL SOLUTIONS

ROUND 1

1. From the identity $(x-y)^2 = x^2-2xy+y^2$, we have that $2013^2+2012^2-2(2012)(2013) = (2013-2012)^2 = \boxed{1}$.

2. First, we have $2♠3 = 5/6$ and $4♠1 = 5/4$, so we must solve the equation $\dfrac{a+5/6}{5a/6} = \dfrac{5}{4}$.

   Cross-multiplying, we have $a = \boxed{20}$.

3. Following order of operations,

$$\begin{aligned} -|-3|^2 + (-1)^{-12} \div \left(\sqrt{1}+1\right) \cdot 2^2 + \frac{6(2+1)}{\sqrt[3]{27}} &= -9+1\div 2\times 4+6 \\ &= -9+\frac{1}{2}\times 4+6 \\ &= -9+2+6 \\ &= \boxed{-1}. \end{aligned}$$

   In particular, note that division and multiplication are on the same level and are evaluated from left to right, so the answer is NOT $-9+\frac{1}{8}+6 = -23/8$.

ROUND 2

1. The average is 80, so the sum is $3(80) = 240$. Therefore the third measurement is $240-64-78 = \boxed{98}$.

2. Find a common denominator for both the numerator and denominator of the large fraction:

$$\frac{\dfrac{p^2+pq-pq+q^2}{q(p+q)}}{\dfrac{pq+q^2+p^2-pq}{p(p+q)}}$$

   Cancel out the common factors of $(p+q)$ in both denominators and cancel out the $+pq$ and $-pq$ in each numerator:

$$\frac{\frac{p^2+q^2}{q}}{\frac{p^2+q^2}{p}}$$

This is equal to $\boxed{p/q}$.

3. Recall the identity $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$. Let $a = x$ and $b = 1/x$ so we have $a + b = 5$ and $ab = 1$.

   Then, $x^3 + \dfrac{1}{x^3} = a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 5^3 - 3(1)(5) = 125 - 15 = \boxed{110}$.

ROUND 3

1. Since 3 and 7 are relatively prime, only multiples of 21 are in the intersection. There are $\boxed{4}$ multiples of 21 between 1 and 100.

2. There are $400-80 = 320$ students comfortable with at least one subject. From inclusion-exclusion, the number of students that feel comfortable with both subjects is $150+200-320 = \boxed{30}$.

3. There are a total of $2^9 = 512$ subsets. The primes are $2, 3, 5, 7$ so five elements are not primes (1 is a unit, neither prime nor composite). Therefore there are $2^9 - 2^5 = 512 - 32 = \boxed{480}$ subsets that contain at least one prime.

ROUND 4

1. The base is a diameter of the semicircle, so it has length 4. The area of a triangle is one-half of the base multiplied by the height. The height is maximized when it equals the radius (2). Hence the area is $\frac{1}{2}(4)(2) = \boxed{4}$.

2. We have

$$\begin{aligned} 5(14+x)^2 &= 14^2(5+x) \\ 5(196+28x+x^2) &= 196(5+x) \\ 140x + 5x^2 &= 196x \\ 5x^2 - 56x &= 0 \end{aligned}$$

   So the answer is $\boxed{56/5}$.

3. The area of the circle is πr^2, so $P = 6$. The area of the equilateral triangle is $r^2/\sqrt{3} = \dfrac{r^2\sqrt{3}}{3}$. The area of the right triangle (a 30-60-90 triangle based on the 1 : 2 ratio of side lengths $DC : DB$) is $\dfrac{r^2\sqrt{3}}{2}$. Hence $Q = -5$. The answer is therefore $P + Q = \boxed{1}$.

ROUND 5

1. The sum of the coefficients of any polynomial p is equal to $p(1)$. Therefore, the sum of the coefficients of $(13x - 15)^5$ is $(13 \cdot 1 - 15)^5 = (-2)^5 = \boxed{-32}$.

2. **METHOD I:** Since k is a constant, the sum of the roots of the polynomial $5x^2 + 34x - k = 0$ is $-34/5$. We are given that one root is -8, so the other root must be $-34/5 - (-8) = \boxed{6/5}$.

   METHOD II: Plug in the root -8 to find $k = 5(-8)^2 + 34(5) = 320 - 272 = 48$. Therefore, the polynomial is $5x^2+34x-48 = 0$ which can be factored as $(x+8)(5x-6) = 0$. Hence the other root is $\boxed{6/5}$, as before.

3. Each root of the new polynomial is 3 less than that of the given polynomial, so this is a change of base from x to $x + 3$. Our desired polynomial is therefore

$$\begin{aligned} 5(x+3)^2 - 13(x+3) + 2012 &= 0 \\ 5(x^2 + 6x + 9) - 13x - 39 + 2012 &= 0 \\ 5x^2 + 30x + 45 - 13x - 39 + 2012 &= 0 \\ 5x^2 + 17x + 2018 &= 0 \end{aligned}$$

   Therefore the answer is $\boxed{5x^2 + 17x + 2018 = 0}$. The fact that the roots of both these polynomials are complex is irrelevant.

TEAM ROUND

1. Note that $x\flat y = (x - y)^2$ and $x\sharp y = (x + y)^2$. Then, the numerator is $(3 - 2)^2\flat(2 + (-1))^2 = 1\flat 1 = (1 - 1)^2 = 0$. We check that the denominator is nonzero (it is 16), so our answer is $\boxed{0}$.

2. With a distance of D miles, we can set the traveling time of each trip calculated separately and the overall trip equal:

$$\frac{D}{42} + \frac{D}{x} = \frac{2D}{48}$$

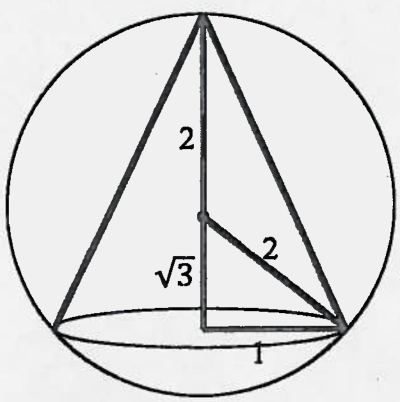
Notice that the quantity D cancels out, so the total distance is irrelevant! (This is intuitively true.) Therefore, we have

$$\frac{1}{42} + \frac{1}{x} = \frac{1}{24}$$

and it follows that $x = \boxed{56}$.

3. We have $A' = \{5,6,7,8,9\}$ and $B' = \{1,3,5,7,9\}$ and $A' \cup B' = \{1,3,5,6,7,8,9\}$. Also, $A \cup B = \{1,2,3,4,6,8\}$. The intersection of these two sets is $\{1,3,6,8\}$. The union with A is $\{1,2,3,4,6,8\}$, so the complement is $\boxed{\{5,7,9\}}$.

4. Use the 30-60-90 right triangle to find lengths:



The volume of the cone is $\frac{1}{3}\pi(1)^2(2+\sqrt{3})$ and the volume of the sphere is $\frac{4}{3}\pi(2)^3$. Hence the ratio is $\boxed{\frac{2+\sqrt{3}}{32}}$.

5. Make the substitution $y = \sqrt{x+1}$. The equation then becomes

$$6y^3 - 11y^2 + 3y = 0.$$

This factors as $y(2y-3)(3y-1) = 0$, so $y = 0, 3/2, 1/3$. Recalling that $y = \sqrt{x+1}$, we have that $\boxed{x = -1, 5/4, -8/9}$.

6. We have that $0.3\overline{6} = \frac{36-3}{90}$ and $0.\overline{15} = \frac{15}{99}$, so the product is $\frac{33}{90} \cdot \frac{15}{99} = \boxed{\frac{1}{18}}$.

7. If a quadrilateral can be inscribed in a circle, it is called a *cyclic quadrilateral* and has the property that opposite angles are supplementary (together, opposite angles intercept the entire circle). Therefore, $180° = m\angle A + m\angle C = (9x + 72)°$ so $x = 12$. We then have $m\angle B = 125°$ so $m\angle D = (180 - 125)° = \boxed{55°}$.

8. Christina's speed is 3/4 of Angela's. Moira's speed is 2/3 of 3/4 of Angela's, or $(2/3)(3/4) = 1/2$ of it. The average of Christina's and Moira's speeds is therefore 5/8 of Angela's, so Angela's speed was $\boxed{8/5}$ of the average of the other two.

9. We note that $22 \equiv 1 \pmod 7$ and $0 \pmod{11}$ while $56 \equiv 0 \pmod 7$ and $1 \pmod{11}$. Therefore, integers that are simultaneously $4 \pmod 7$ and $2 \pmod{11}$ are $4 \cdot 22 + 2 \cdot 56 \pmod{77}$. Since $4 \cdot 22 + 2 \cdot 56 = 200$, all solutions are equivalent to $200 \pmod{77}$. The smallest positive solution is therefore $200 - 2(77) = \boxed{46}$. [For more information, read about the CHINESE REMAINDER THEOREM]